Algebraic Aspects of the Theory of Product Structures in Complex Cobordism

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February 1, 2008

Abstract

We address the general classification problem of all stable associative product structures in the complex cobordism theory. We show how to reduce this problem to the algebraic one in terms of the Hopf algebra S (the Landweber-Novikov algebra) acting on its dual Hopf algebra S^* with a distinguished "topologically integral" part Λ that coincids with the coefficient ring of the complex cobordism. We describe the formal group and its logarithm in terms of representations of S. We introduce one-dimensional representations of a Hopf algebra. We give series of examples of such representations motivated by well-known topological and algebraic results. We define and study the divided difference operators on an integral domain. We discuss certain important examples of such operators arising from analysis, representation theory, and noncommutative algebra. We give a special attention to the division operators by a noninvertible element of a ring. We give new general constructions of associative product structures (not necessarily commutative) using the divided difference operators. As application, we describe new classes of associative products in the complex cobordism theory.

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1 Introduction. Complex cobordism theory and algebra

Complex cobordism (bordism) functor $U^*(X)$, $U^*(X,Y)$ is a well-known generalized cohomology (homology) theory with dual elements $u \in U_k(X)$ being represented by maps $f : M^k \to X$ of closed U-manifolds to X (a complex structure is given in the normal bundle of some embedding $M^k \subset \mathbf{R}^{n+k}$, see [1]). Algebraically, $U^*(X,Y)$ is a commutative and associative $\mathbb{Z}$-graded ring with a unit $1 \in U^*(pt) = \Omega_U^* = \Lambda$ (for any pairs (X,Y)). According to the results of Milnor and Novikov (see [2]-[4]), this ring is isomorphic to the polynomial ring $\Omega_U^* = \mathbb{Z}[u_2, u_4, u_6, \dots]$ with even-dimensional generators, $\deg u_k = -2k$.

Problem. Give classification of all stable associative product structures in the complex cobordism theory (on the category of finite CW-complexes).

To make it precise, a "product structure in the complex cobordism" is a bilinear operation $u \circ v = \Phi(u \otimes v)$ where

$$\Phi : U^*(X) \otimes U^*(X) \to U^*(X),$$

defined for all spaces X, all $u, v \in U^*(X)$, and commuting with continuous maps $f : X \to Y$, i.e.

$$f^*(u) \circ f^*(v) = \Phi(f^*(u) \otimes f^*(v)) = f^*\Phi(u \otimes v) = f^*(u \circ v).$$

A product structure is *stable* if it commutes with the suspension isomorphism s. Recall that there is a distinguished element $\mu \in U^1(S^1)$ such that the suspension isomorphism $s : U^*(X) \to \widetilde{U}^*(\Sigma X)$ is given by $u \to \mu u = s(u)$. Here S^1 is a circle, $\Sigma X = S^1 \wedge X$ is a suspension over X, and $\widetilde{U}^*(\Sigma X) = U^*(\Sigma X, pt)$. Here and below the symbol $\wedge$ denotes the smash-product in the category of spaces with a base point, i.e. $X \wedge Y = X \times Y / X \vee Y$. Thus a new product structure $\Phi(u \otimes v) = u \circ v$ is stable if the condition

$$\mu(u \circ v) = (\mu u) \circ v = (-1)^{q_1} u \circ (\mu v) \tag{1}$$

is satisfied. Here $q_1 = \deg u$ and uv denotes the usual standard product in the complex cobordism. Clearly the standard product is stable.

We recall that the complex cobordism theory is a module over the "Steenrod algebra" of all operations A^U, i.e. the linear operators $a : U^*(X) \to U^*(X)$ commuting with all continuous maps and the suspension isomorphism. This algebra has been computed in [5] and consists of two parts:

1. The coefficient ring $\Lambda = \Omega^U \subset A^U$ that consists of the operations of multiplication by a "scalar" $\lambda \in \Lambda$: $u \to \lambda u$.

2. The Landweber-Novikov algebra S (see [5], [6]). This is a Hopf algebra over $\mathbb{Z}$, it is $\mathbb{Z}^+$-graded, and $S = \sum_{j \geq 0} S^j$ with a basis $s_w \in S$ where $S^0 = \mathbb{Z}$. Here $w = (k_1, \ldots, k_l)$ is either an unordered collection of positive integers $k_j \in \mathbb{Z}^+$, or the empty set with $s_\emptyset = s_0 = 1 \in S^0$ and $\deg w = 2\sum k_q$.

The coproduct (or diagonal) in S is given by

$$\Delta s_w = \sum_{(w', w'') = w} s_{w'} \otimes s_{w''}. \tag{2}$$

Furthermore, the Hopf algebra S acts on all $U^*(X)$ (in particular on the coefficient ring $\Lambda = U^*(pt) = \Omega^U$) providing $U^*(X)$ with a structure of "Milnor module":

$$s_w(uv) = \sum_{(w', w'') = w} s_{w'}(u) s_{w''}(v). \tag{3}$$

The elements $s_{(n)}$ are *primitive* in S and they generate a Lie algebra; indeed, one has the identities $[s_{(n)}, s_{(m)}] = (m - n)s_{(m+n)}$.

An element $x \in U^2(X)$ is a *geometric cobordism element* if the following conditions are satisfied

$$s_{(k)}(x) = x^{k+1} \quad \text{and} \quad s_w(x) = 0 \quad \text{if} \quad w = (k_1, \ldots, k_l), \quad l > 1. \tag{4}$$

There is a natural representation

$$s \to s(x_1 x_2 \cdots x_N), \quad s \in S,$$

of the algebra S (and the whole algebra A^U) on the geometric cobordism elements. This representation is *asymptotically faithful* in the sense that for any element $s \in S$ there exists an integer $N > m$ so that $s(x_1 x_2 \cdots x_N) = 0$ is equivalent to $s = 0$. Recall also that the representation $\lambda \to s\lambda$ of S on the coefficient ring Λ is faithful. The latter was computed in [5]. The paper [7] gives interesting interpretation of this representation in terms of differential operators on infinite dimensional Lie group of formal derivations on the real line. This example leads to the theory of operators on quantum groups (see [8], [9]). The algebra A^U of operations in complex cobordism may be described as the completion $A^U = (\Lambda S)^\wedge$, i.e. its elements are the series

$$\sum_{j=0}^{\infty} \lambda_j s_{w_j} \in A^U, \quad \lambda_j \in \Lambda, \quad s_{w_j} \in S,$$

where $\deg w_j \to \infty$ while $j \to \infty$. The commutation relations in A^U are determined by the representation of the algebra S on the coefficient ring Λ.

Let S^* be the Hopf algebra dual to S. Then there is the embedding $\Lambda \subset S^*$ given by $\lambda(s_w) = \varepsilon s_w(\lambda)$ where $\lambda \in \Lambda$ and $\varepsilon : \Lambda \to \mathbb{Z}$ is the augmentation ($\varepsilon(1) = 1$ and $\varepsilon(\lambda) = 0$ if $\deg \lambda < 0$). Clearly $\Lambda \otimes Q = S^* \otimes Q$ and S^* is the polynomial ring over $\mathbb{Z}$ on generators s_k^* such that $(s_k^*, s_w) = 1$ if $w = (k)$ and $(s_k^*, s_w) = 0$ otherwise. Below we give an algebraic description of the subring $\Lambda \subset S^*$.

Lemma 1. *Any stable product structure Φ in the complex cobordism is given by a formal series*

$$\widehat{\Phi} = \sum \lambda_{ij} s_{w_i} \otimes s_{w_j} \tag{5}$$

where $\lambda_{ij} \in \Lambda$, $\deg w_i \to \infty$, $\deg w_j \to \infty$, while $i, j \to \infty$, so that the product structure Φ is given by

$$u \circ v = \Phi(u \otimes v) = \sum \lambda_{ij} s_{w_i}(u) s_{w_j}(v) \tag{6}$$

for any $u, v \in U^*(X)$.

Furthermore, the coefficients of the series $\widehat{\Phi}$ are uniquely determined by the products $u \circ v$ for $u, v \in \Lambda = U^*(pt) = \Omega^U$.

Proof. Let $x(N) \in U^{2N}(MU_N)$ be a canonical Thom class corresponding to the universal complex bundle $\eta_N \to BU(N)$ where $\dim \eta_N = N$ and MU_N is the Thom space of this bundle.

It is well-known that $U^*(MU_N, pt)$ is one-dimensional A^U-module generated by $x(N)$ and the elements $s_w x(N)$ with $w = (k_1, \ldots, k_l)$, $l \leq N$, $x(N) = s_0 x(N)$, form its basis over Λ.

Let Φ be a stable product structure in the complex cobordism theory. Then the products $x(N_1) \circ x(N_2) \in U^*(MU_{N_1} \times MU_{N_2})$ are defined for any N_1, N_2. The bilinearity of Φ implies

$$\Phi(x_{N_1}, x_{N_2}) = x(N_1) \circ x(N_2) \in \widetilde{U}^*(MU_{N_1} \wedge MU_{N_2}).$$

Thus this product may be uniquely written as a formal series

$$x(N_1) \circ x(N_2) = \sum \lambda_{ij} s_{w_i}(x(N_1)) s_{w_j}(x(N_2))$$

where $w_i = (k_1', \ldots, k_{l_1}')$, $l_1 \leq N_1$, and $w_j = (k_1'', \ldots, k_{l_2}'')$, $l_2 \leq N_2$. The coefficients $\lambda_{ij} \in \Lambda$ do not depend on N_1 and N_2 since the product is stable and commutes with continuous maps. We recall here that the Thom spaces MU_N for different N are related to each other by means of the canonical maps $e_{N_1,N_2} : \Sigma^{2(N_2-N_1)} MU_{N_1} \to MU_{N_2}$ such that $e_{N_1,N_2}^* x(N_2) = s^{2(N_2-N_1)} x(N_1)$.

Let $u \in U^{q_1}(X, Y)$ and $v \in U^{q_2}(X, Y)$ be any two elements. Then there exist maps

$$f_1 : \Sigma^{2N_1 - q_1} X/Y \to MU_{N_1} \quad \text{and} \quad f_2 : \Sigma^{2N_2 - q_2} X/Y \to MU_{N_2}$$

representing u and v for some N_1 and N_2, i.e. $f_1^* x(N_1) = s^{2N_1 - q_1} u$, $f_2^* x(N_2) = s^{2N_2 - q_2} v$. We define a map

$$f = f_1 \wedge f_2 : \Sigma^{2(N_1+N_2)-(q_1+q_2)} X \xrightarrow{f_1 \wedge f_2} MU_{N_1} \wedge MU_{N_2}.$$

Then we have

$$\begin{aligned} f^*(x(N_1) \circ x(N_2)) &= \sum \lambda_{ij} s_{w_i}(s^{2N_1-q_1} u) s_{w_j}(s^{2N_2-q_2} v) \\ &= (-1)^{q_1 q_2} s^{2(N_1+N_2)-(q_1+q_2)} \Big(\sum \lambda_{ij} s_{w_i}(u) s_{w_j}(v) \Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} f^*(x(N_1) \circ x(N_2)) &= (f_1^* x(N_1)) \circ (f_2^* x(N_2)) = (s^{2N_1-q_1} u) \circ (s^{2N_2-q_2} v) \\ &= (-1)^{q_1 q_2} s^{2(N_1+N_2)-(q_1+q_2)} u \circ v. \end{aligned}$$

This proves the first statement of Lemma 1.

Now let $u \circ v$ be a product on the coefficient ring $\Lambda = U^*(pt)$. We notice that if $u \in \Lambda^{-2n}$ and $v \in \Lambda^{-2m}$ then

$$u \circ v = \Phi(u \otimes v) = \sum_{\substack{\deg w_i \leq 2n, \\ \deg w_j \leq 2m}} \lambda_{ij} s_{w_i}(u) s_{w_j}(v). \tag{7}$$

In particular, for $\deg w = 2n$ we have

$$s_w(u) = \varepsilon s_w(u) = \langle u, s_w \rangle$$

where $\varepsilon : \Lambda \to Z$ is the augmentation.

Now we use (7), the above embedding $\Lambda \subset S^*$, and the isomorphism $\Lambda \otimes Q = S^* \otimes Q$ to recover the coefficients λ_{ij} by induction on $\deg w$. Since we use only the product $u \circ v$ on Λ and the action of algebra S on Λ, this completes the proof of the lemma. $\square$

Lemma 1 allows us to define a new product in the complex cobordism $U^*(X)$ as a series $\widehat{\Phi}$ given by (5). Below we study properties of such products.

Now we describe this situation in terms of Hopf algebras. Suppose S is a Hopf algebra and Λ is a Milnor module over S (i.e. S acts on Λ and $s(uv) = \sum s_i'(u)s_i''(v)$ where $\Delta(s) = \sum s_i' \otimes s_i''$). Then the new product structure is defined by (6). This gives new setting for the classification problem of product structures.

The most interesting case arising from the complex cobordism theory is when the Milnor module is S^*, that is the Hopf algebra dual to S in the basis s_w with $\Lambda \subset S^*$. The action of S on S^* is given by $s(u) = R_s^*(u)$ where R_s is the *right multiplication operator* (see [7], [8]):

$$R_s(s') = s's, \quad R_s : S \to S, \quad R_s^* : S^* \to S^*, \quad (R_s^*(u), s') = (u, s's). \tag{8}$$

Thus the classification problem of product structures in complex cobordism $U^*(X)$ reduces to study of the single Hopf algebra S acting (by means of R^*) on its dual Hopf algebra S^* with a distinguished "topologically integral" part $\Lambda \subset S^*$.

How to describe the ring $\Lambda \subset S^*$ in algebraic terms? For this we use geometric cobordism elements $x \in U^2(X)$ defined above. Consider the Milnor module $S^*[[x_1, x_2]]$ of formal series on x_1, x_2 over S^*. Define an action of S on this module algebraically, provided x_1 and x_2 are geometric cobordism elements, by

$$\begin{gathered} s_{(k)}(x_j) = x_j^{k+1}, \qquad s_w(x_j) = 0, \quad w = (k_1, \dots, k_l), \quad l > 1, \\ s(\lambda) = R_s^*(\lambda), \qquad (R_s^*(\lambda), s') = (\lambda, s's), \\ s(uv) = \sum s_i'(u)s_i''(v), \qquad \Delta = \sum s_i' \otimes s_i''. \end{gathered}$$

This determines the action of S on $S^*[[x_1, x_2]]$.

Lemma 2. *There exists a unique series*

$$x = f(x_1, x_2) = x_1 + x_2 + \sum_{i,j \geq 1} \alpha_{ij} x_1^i x_2^j, \qquad \alpha_{ij} \in S^*, \tag{9}$$

such that $x = f(x_1, x_2)$ is a geometric cobordism element provided x_1 and x_2 are geometric cobordism elements. In particular, the action of S on x is determined by (4).

Proof. We define the operator $S_t = \sum_{w \geq 0} s_w t^w$ with $w = (k_1, \dots, k_l)$, $t^w = t_1^{k_1} \cdots t_l^{k_l}$, where t_i, $i = 1, 2, \dots$, are algebraically independent elements. It is easy to see that

$$\Delta S_t = \sum_{w \geq 0} (\Delta s_w) t^w = S_t \otimes S_t \quad \text{or} \quad S_t(uv) = S_t(u)S_t(v).$$

Let us compute $S_t(x)$ for any geometric cobordism element x. By definition of geometric cobordism elements, $S_t(x)$ is equal to certain series $\phi_t(x) = \sum_{k=0}^{\infty} x^{k+1} t_k$ with $t_0 = 1$. Now we apply the operator S_t to the both sides of (9).

$$S_t(x) = S_t(x_1) + S_t(x_2) + \sum_{i,j \geq 1} S_t(\alpha_{ij}) S_t(x_1)^i S_t(x_2)^j \tag{10}$$

where x, x_1, x_2 are geometric cobordism elements, i.e. $S_t(x) = \phi_t(x)$, $S_t(x_1) = \phi_t(x_1)$, $S_t(x_2) = \phi_t(x_2)$. Now we apply the augmentation ε to (10):

$$\varepsilon : S^* \to \mathbb{Z}$$

where $\varepsilon(s_n^*) = 0$ if $n \neq 0$, $S^* = \sum_{n \leq 0} S_n^*$, and $\varepsilon(x_1) = x_1$, $\varepsilon(x_2) = x_2$. We obtain

$$\begin{aligned} \varepsilon S_t(x) &= \sum \varepsilon(x)^{k+1} t_k, \qquad \varepsilon(x) = x_1 + x_2, \\ \varepsilon S_t(x_1) &= \sum \varepsilon(x_1)^{k+1} t_k = \sum x_1^{k+1} t_k, \\ \varepsilon S_t(x_2) &= \sum \varepsilon(x_2)^{k+1} t_k = \sum x_2^{k+1} t_k. \end{aligned}$$

The following formula

$$\varepsilon S_t(\alpha_{ij}) = \sum_w \varepsilon s_w(\alpha_{ij}) t^w = \sum_w (s_w, \alpha_{ij}) t^w$$

follows from definitions. It implies

$$\varepsilon S_t(x) = \phi_t(x_1 + x_2) = \phi_t(x_1) + \phi_t(x_2) + \sum_{i,j} \sum_w (s_w, \alpha_{ij}) t^w \phi_t(x_1)^i \phi_t(x_2)^j.$$

Let $y_j = \phi_t(x_j)$. Then $x_j = \phi_t^{-1}(y_j)$ and

$$\varepsilon S_t(x) = \phi_t(\phi_t^{-1}(y_1) + \phi_t^{-1}(y_2)) = \sum_{i,j} \sum_w (s_w, \alpha_{ij}) t^w y_1^i y_2^j.$$

Compare the coefficients of the terms $y_1^i y_2^j$ in the left and right hand sides. Since (s_w, α_{ij}) are already known on the left-hand side, α_{ij} are uniquely determined as elements of S^*, i.e. as linear forms on S. This proves the uniqueness.

The existence of the series (9) follows from well-known results on the complex cobordism theory (see [5]). An algebraic proof of its existence follows from Lemma 3 below. This completes the proof of Lemma 2. □

Lemma 2 implies the following.

Corollary 1. *The series $x = f(x_1, x_2)$ determines a formal group on the set of geometric cobordism elements (see* [9]*).*

Definition 1. The subring $\Lambda \subset S^*$ generated over $\mathbb{Z}$ by the elements $\alpha_{ij} \in S^*$ is called the complex cobordism ring. (It is well-known that Λ is a polynomial ring, see above.)

Remark 1. A formal group in complex cobordism theory was defined in [5] geometrically. It coincides with the one given by the series (9). We emphasize that we give here entirely algebraic definition of this formal group and the complex cobordism ring Λ using only the Hopf algebra S.

The formal group $f(x_1, x_2)$ over the ring $\Lambda \otimes \mathbb{Q}$ is given by

$$f(x_1, x_2) = g^{-1}(g(x_1) + g(x_2)),$$

where $g^{-1}(g(x)) = x$ and $g(x) = x + \sum b_i x^{i+1}$, $b_i \in \Lambda \otimes \mathbb{Q}$. The series $g(x) = g_f(x)$ is called the *logarithm of the formal group* $f(x_1, x_2)$. We would like to describe the logarithm $g(x)$ again by means of the Hopf algebra S.

Lemma 3. *There exists a unique series*

$$g(x) = x + \sum_{i=1}^{\infty} b_i x^{i+1}, \qquad b_i \in S^*,$$

such that $s_w g(x) = 0$ *for any geometric cobordism element* x *and for all* w *with* $\deg w > 0$. *This series is a logarithm of the formal group and satisfies the condition*

$$x = g(x) + \sum_{k=1}^{\infty} s_k^* g(x)^{k+1},$$

i.e. $g^{-1}(t) = t + \sum_{k=1}^{\infty} s_k^* t^{k+1}$ *where* s_k^* *are multiplicative generators of* S^* *dual to* $s_{(k)} \in S$ *(with respect to the* $\mathbb{Z}$*-basis* $\{s_w\}$*).*

Proof. Let x be a geometric cobordism element and let $g(x) = x + \sum b_i x^{i+1}$ ($b_i \in S^*$) be a series such that $s_w g(x) = 0$ for all w with $w \neq \emptyset$. Then $S_t g(x) = g(x)$ where $S_t = \sum s_w t^w$. It implies that $x = g^{-1}(g(x))$. Thus $\phi_t(x) = S_t x = S_t[g^{-1}](S_t g(x)) = S_t[g^{-1}](g(x))$ where if $g^{-1}(t) = t + \sum a_i t^{i+1}$, $a_i \in S^*$ then $S_t[g^{-1}] = t + \sum S_t(a_i) t^{i+1}$. Now let x_1 and x_2 be geometric cobordism elements. Consider the series

$$F(x_1, x_2) = g^{-1}(g(x_1) + g(x_2)) = x_1 + x_2 + \sum \alpha'_{ij} x_1^i x_2^j.$$

We have

$$g(x_1) + g(x_2) = g(F(x_1, x_2)).$$

Then

$$\begin{aligned} S_t F(x_1, x_2) &= S_t[g^{-1}](S_t g(x_1) + S_t g(x_2)) \\ &= S_t[g^{-1}](g(x_1) + g(x_2)) = S_t[g^{-1}](g(F(x_1, x_2))). \end{aligned}$$

We use the identity $S_t[g^{-1}](g(x)) = \phi_t(x)$ to obtain

$$S_t F(x_1, x_2) = \phi_t(F(x_1, x_2)).$$

We see that the series $F(x_1, x_2)$ is a geometric cobordism element whence according to Lemma 2, it coincides with the series (9). Thus if a series $g(x)$ satisfies the conditions of Lemma 3 then it is unique and coincides with the logarithm of the formal group (9).

Let us give an algebraic proof of the existence of such series. First, we consider $S^*[[t]]$ as a Milnor module with the following action of the algebra S:

$$s_0 t = t, \quad s_w t = 0, \quad \deg w > 0 \quad \text{and} \quad s_w(\lambda) = R_s^*(\lambda), \quad \lambda \in S^*.$$

We show that the series

$$x = \gamma(t) = t + \sum s^*_{(n)} t^{n+1} \in S^*[[t]]$$

is a geometric cobordism element. It is enough to prove that

$$s_w(s^*_{(n)}) = \begin{cases} 0, & \text{if } w \neq (k), \\ \gamma_{n,k}, & \text{if } w = (k), \end{cases}$$

where $\gamma_{n,k}$ are the coefficients of t^{n+1} in the series $\gamma(t)^{k+1}$. To complete the argument we use the formula

$$\langle s_{w_2}(s^*_n), s_{w_1} \rangle = \langle s^*_n, s_{w_1} s_{w_2} \rangle$$

and the fact that the representation of S on a product of geometric cobordism elements is asymptotically faithful.

Now let $g(x) \in S^*[[t]]$ be such that $g(x) = g(\gamma(t)) = t$. Then x is a geometric cobordism element and by construction $s_w(t) = 0$ if $\deg w > 0$. Thus the series $g(x)$ satisfies all conditions given in Lemma 3. This completes the proof. □

We remark that the result of Lemma 3 has been proven first by means of the complex cobordism theory (see [10]).

Definition 2. We call a representation of a Hopf algebra S on a Milnor module P one-dimensional if P is the polynomial algebra (or the algebra of formal series) with one generator $u \in P$ over some coefficient ring. The algebra S may act nontrivially on the coefficient ring (which is $\mathbb{Z}$, $\mathbb{Q}$, Λ in our examples, or some other Milnor submodule in $S^* \otimes \mathbb{Q}$). A *height of such representation* is a minimal number k so that $u^{k+1} = 0$.

For instance, P may be a polynomial algebra with one generator $u \in P$ over the ring $\Lambda \subset S^*$ where S acts as it was described above.

Examples. If $k = 1$ we can let $P = U^*(S^{2q})$ for any q. Already for $k = 2$, one-dimensional representations over $\mathbb{Z}$ are known only for $\dim u = 2, 4$ or 8. For $k \geq 3$, we know examples from topology only for the cases $\dim u = 2$ or 4 and $P = U^*(\mathbb{C}P^k)$ or $P = U^*(\mathbb{H}P^k)$ respectively. The case $k = \infty$ is of special interest. For instance, over $\mathbb{Q}$ there are one-dimensional representations of S on $P = U^*(\Omega S^{2q+1}) \otimes \mathbb{Q} = \Omega_U \otimes \mathbb{Q}[[u]]$ with $\dim u_q = 2q$ for all q.

Consider the case $q = 1$ in more detail. A generator of the cohomology group $H^2(\Omega S^3, \mathbb{Z}) = \mathbb{Z}$ is represented by a map $\phi : \Omega S^3 \to \mathbb{C}P^\infty$. The map ϕ gives a geometric cobordism element $u_1 = \phi^*(u) \in U^2(\Omega S^3)$ where $u \in U^2(\mathbb{C}P^\infty)$ is a canonical geometric cobordism element. We have the isomorphism $U^*(\Omega S^3) \otimes \mathbb{Q} = \Omega_U \otimes \mathbb{Q}[[u_1]]$. It is well-known that there exists a map $\psi : \Sigma\Omega S^3 \to S^3$ giving an element $\beta \in U^2(\Omega S^3)$ such that $s\beta = \psi^*\alpha$ where α is a generator of the group $U^3(S^3) = \mathbb{Z}$ and s is the suspension isomorphism. All operations s_w are stable whence $s_w\alpha = 0$ for $\deg w > 0$ implies $s_w\beta = 0$ for $\deg w > 0$. The element $\beta \in U^*(\Omega S^3) \otimes \mathbb{Q}$ is given by a series $\beta = g(u_1)$ where u_1 is a geometric cobordism element and $g(u_1) = u_1 + \cdots$. Thus Lemma 3 implies that $g(u_1)$ coincides with the logarithm of the above formal group and $g^{-1}(\beta) = \beta + \sum_{k \geq 1} s^*_k \beta^{k+1} = u_1 \in U^2(\Omega S^3)$. We emphasize that the elements

$s_k^* \in S^*$ do not belong to the subring $\Lambda \subset S^*$, only their multiples are: $(k+1)!s_k^* \in \Lambda$ (see [10]).

To describe one-dimensional representations of S on $P = U^*(\Omega S^{2q+1}) \otimes \mathbb{Q}$ (in the case $q > 1$) one should use the canonical maps

$$\phi_q : \Omega S^{2q+1} \to \Omega SU(2q+1) \to BU$$

representing the corresponding generators of the homotopy group $\pi_{2q+1}(SU(2q+1)) = \mathbb{Z}$. Here $SU(2q+1)$ is a special unitary group and BU is the infinite dimensional complex Grassmannian. In order to construct such maps one uses that the Bott periodicity implies $BU \approx \Omega SU(\infty)$.

There is an important example of one-dimensional (graded) $\mathbb{Z}$-representation of the Hopf algebra S given by a geometric cobordism elements $u = x$ (as above) with $\dim u = 2$. One more example of one-dimensional Λ-representation of S is given by the elements $u = x\bar{x}$, $\dim u = 4$, where $\bar{x}$ is the element inverse to x in the formal group. Here $s_w(u) \in \Lambda[[u]]$. Such elements u generate a two-valued formal group (this was first observed in [11]). The algebraic theory of two-valued formal groups was developed in [12]. The case $\dim u = 8$, $k = 2$ is also very interesting. Here $P = U^*(CaP^2)$ where CaP^2 is the projective plane over the Kelley numbers. We suspect that there are no one-dimensional representations of S for $k \geq 3$ with $\dim u > 4$. It would be useful to prove this.

Problem. Give classification of one-dimensional $\mathbb{Z}$-representations and Λ-representations of the Hopf algebra S. The representations should preserve grading under the automorphisms

$$u = \Psi(v) = v + \sum_{i \geq 1} \lambda_i v^{i+1}$$

where $\lambda_i \in \mathbb{Z}$ (or $\mathbb{Q}$), or $\lambda_i \in \Lambda$ (or $\lambda_i \in \Lambda \otimes \mathbb{Q} = S^* \otimes \mathbb{Q}$).

Some examples of such representations over certain subrings of $\Lambda \otimes \mathbb{Q}$ may be found in [13]. They are given by the elements $u_n = x[x]_{\varepsilon_n} \cdots [x]_{\varepsilon_n^{n-1}}$ with $\deg w = 2n$. Here ε_n is the nth root of unity and $[x]_{\varepsilon_n} = g^{-1}(\varepsilon_n g(x))$ is a geometric cobordism element that is nothing but the ε_n-th power in the formal group $f(x_1, x_2)$ of the canonical geometric cobordism element x. According to [13], $u_n \in \Lambda_{\{n\}}[[x]]$ where $\Lambda_{\{n\}} = \Lambda \otimes \mathbb{Z}_{\{n\}}$. Here

$$\mathbb{Z}_{\{n\}} = \left\{ \frac{m}{d} \in \mathbb{Q} \;\middle|\; \begin{array}{c} (d,p) = 1 \text{ if the equation } x^n = 1 \text{ has precisely } n \\ \text{distinct solutions in the ring of } p\text{-adic numbers} \end{array} \right\} \subset \mathbb{Q}.$$

For instance, one has $\mathbb{Z}_{\{2\}} = \mathbb{Z}$. This case gives Λ-representations for $u = x\bar{x}$ (see above). These Λ-representations are related to the theory of two-valued formal groups. Furthermore, one-dimensional $\Lambda_{\{n\}}$-representations of S on $P = \Lambda_{\{n\}}[[u_n]]$ with $\deg u_n = 2n$ are related to n-valued formal groups. The existence of n-valued formal groups was observed first in [11]. First results on the algebraic theory of these groups are given in [14]. The contemporary state of affairs in the theory of multi-valued formal groups may be found in [15].

There is another useful example of one-dimensional $\mathbb{Z}$-representation of the Hopf algebra S with $\dim u = 2$. This representation is not equivalent to any action on a

geometric cobordism element (see [7]). Here is the action:

$$s_{(1)}(u) = u^2, \qquad s_{(n)}(u) = 0, \quad n \geq 2. \tag{11}$$

We recall that one has $s_{(n)}(u) = u^{n+1}$ on a geometric cobordism element u (see (4) above). It is easy to see that the representation (11) of S on $P = \mathbb{Z}[u]$ is given by the differential operators

$$s_{(1)} \to u^2 \frac{d}{du}, \qquad s_{(n)} \to 0, \quad n \geq 2.$$

Furthermore, the image of the Hopf algebra S (in the algebra of differential operators) is generated by $\frac{1}{n!}(u^2 d/du)^n$.

There is an interesting class of such representations where P is a polynomial algebra on one generator with negative degree, i.e. $P = \mathbb{Z}[u]$ ($\mathbb{Q}[u]$) or $P = \Lambda[u]$ ($\Lambda \otimes \mathbb{Q}[u]$). Here $\dim u = -2q < 0$.

Example. Let $u \in \Lambda = \Omega^U$ be an element with $\dim u = -2$ or -4. If $u =\in \Lambda^{-2}$ let $u = [\mathbb{C}P^1]$ and if $u \in \Lambda^{-4}$ let $u = 3[\mathbb{C}P^1]^2 - 4[\mathbb{C}P^2]$. Notice that in the latter case $s_{(1)}(u) = 0$. In both cases the action of S on u is determined by the operations $s_w(u) \in \mathbb{Z}$ with $\deg w = 2$. Thus we have one-dimensional $\mathbb{Z}$-representations of the algebra S.

2 Division operators and product operators

Here we develop an algebraic machinery which allows us to produce examples of stable product structures in the complex cobordism theory. We will carry on all computations in the category of modules over some commutative associative ring K (the "scalars") with the unit $1 \in K$. Furthermore, all modules R in our category are also commutative and associative rings with the unit $1 \in K \subset R$. We also assume that K and R are integral domains. All operators below are assumed to be K-linear.

Definition 3. A linear operator $\partial : R \to R$ is called a *divided difference operator* if it is not identically trivial and satisfies the following identity

$$\partial(xy) = (\partial x)y + x(\partial y) - \alpha(\partial x)(\partial y) \tag{12}$$

for all $x, y \in R$, where $\alpha \in R$ is not invertible in R.

It follows from (12) that $\partial(1) = 0$ for any divided difference operator ∂ (under our assumptions).

An operator $\pi : R \to R$ is *multiplicative* if $\pi(xy) = \pi(x)\pi(y)$.

Lemma 4. *An operator ∂ is a divided difference operator if and only if the operator $\pi = 1 - \alpha\partial$ is multiplicative.*

Proof. Let ∂ be a divided difference operator. Then the definition gives:

$$\begin{aligned} \pi(xy) &= xy - \alpha\partial(xy) = xy - \alpha(\partial x)y - \alpha x(\partial y) + \alpha^2(\partial x)(\partial y) \\ &= (x - \alpha\partial x)(y - \alpha\partial y) = \pi(x)\pi(y). \end{aligned}$$

It is easy to see that the converse statement holds as well. This proves Lemma 4. $\square$

Lemma 4 motivates the term "divided difference operator".

Lemma 5. *Let ∂ be a divided difference operator such that $\partial^2 = \gamma\partial$ with $\gamma \in \mathbb{R}$. Then $(1-\alpha\gamma)\partial(\alpha) = 2 - \alpha\gamma$. In particular, $\partial(\alpha) = 2$ if $\partial^2 = 0$.*

Proof. Indeed, the definition gives:

$$\begin{aligned}\partial^2(xy) &= \partial\big((\partial x)y + x(\partial y) - \alpha(\partial x)(\partial y)\big)\\ &= (\partial^2 x)y + (\partial x)(\partial y) - \alpha(\partial^2 x)(\partial y) + (\partial x)(\partial y) + x(\partial^2 y) - \alpha(\partial x)(\partial^2 y)\\ &\quad -(\partial\alpha)(\partial x)(\partial y) - \alpha\partial\big((\partial x)(\partial y)\big) + \alpha(\partial\alpha)\partial\big((\partial x)(\partial y)\big).\end{aligned}$$

The condition $\partial^2 = \gamma\partial$ implies the identity

$$(1-\alpha\gamma)[(1-\alpha\gamma)\partial(\alpha) - (2-\alpha\gamma)]\partial x\partial y = 0.$$

We recall that ∂ is not identically trivial, R is an integral domain, and α is not invertible in R. This proves the result. □

Lemma 6. *A divided difference operator ∂ satisfies the condition $\partial^2 = \gamma\partial$ if and only if $\pi^2 = 1$ and $\pi(\alpha) = -\alpha/(1-\alpha\gamma)$ where $\pi = 1 - \alpha\partial$.*

Proof. Lemma 4 asserts that π is a multiplicative operator. Thus

$$\begin{aligned}\pi^2 x &= \pi(x - \alpha\partial x) = \pi(x) - \pi(\alpha)\pi(\partial x)\\ &= x - \alpha\partial x - \pi(\alpha)\partial x + \alpha\pi(\alpha)\partial^2 x = x - (\alpha + \pi(\alpha))\partial x + \alpha\pi(\alpha)\partial^2 x.\end{aligned} \tag{13}$$

Let $\partial^2 = \gamma\partial$. Then Lemma 5 gives $\partial(\alpha) = (2-\alpha\gamma)/(1-\alpha\gamma)$ whence $\pi(\alpha) = \alpha - \alpha\partial\alpha = -\alpha/(1-\alpha\gamma)$. Thus the condition $\partial^2 = \gamma\partial$ implies that $\pi^2 x = x$ for all x. Conversely, the conditions $\pi^2 = 1$ and $\pi(\alpha) = -\alpha/(1-\alpha\gamma)$ together with (13) imply $\partial^2 = \gamma\partial$. This proves Lemma 6. □

The constructions given here suggest an interpretation of R as a space of functions on some space and operator π as a translation operator of that space or another homeomorphism of this space. Of course this interpretation makes sense if the operator π is invertible. In important examples, π turns out to be a projector, i.e. $\pi^2 = \pi$. In order to construct multiplicative projectors, the "division operators" (i.e. operators dividing by a given scalar) were used in [5, p.887]. These operators are defined by the identity (12) and the requirement $\partial(\alpha) = 1$. The paper [16] gives a classification of such operators, in particular, it was shown (see Lemma 12 below) that the well-known Adams-Quillen projectors are compositions of the Novikov's division operators by some scalars α (under a special choice of these scalars).

Lemma 7. *A divided difference operator ∂ is a division operator by a scalar α if and only if the composition $\partial\alpha = \partial \circ \alpha$ is the identity operator, in particular,*

$$\pi(x) = (\partial\alpha - \alpha\partial)(x) = [\partial, \alpha](x).$$

Proof. The definitions imply

$$\partial\alpha(x) = \partial(\alpha x) = (\partial(\alpha))x + \alpha(\partial x) - \alpha(\partial(\alpha))(\partial x).$$

If $\partial(\alpha) = 1$ then $\partial\alpha x = x + \alpha(\partial x) - \alpha(\partial x) = x$. Conversely, let $\partial(\alpha) = 1$, i.e. $\partial(\alpha x) = x$. This implies

$$\partial(\alpha x) = (\partial(\alpha))x + \alpha(\partial x) - \alpha\partial(\alpha)(\partial x) = x.$$

Thus

$$(1 - \partial(\alpha))x = \alpha(\partial x)(1 - \partial(\alpha)),$$

or

$$(1 - \partial(\alpha))[x - \alpha(\partial x)] = 0. \tag{14}$$

Now we let $x = 1$ in (14). Then $\partial(1) = 0$ gives $\partial(\alpha) = 1$. The result follows. $\square$

We assume now that the ring R satisfies the following condition. If $\alpha \in R$ is a noninvertible element then $\bigcap_n(\alpha^n R) = 0$ where $(\alpha R) \subset R$ is the principal ideal generated by α. This assumption holds in all our examples.

Lemma 8. *Let ∂ be a divided difference operator. Then the kernel of $\pi = 1 - \alpha\partial$ is nontrivial if and only if ∂ is a division operator.*

Proof. Let ∂ be a division operator, i.e. $\partial(\alpha) = 1$. Then $\pi(\alpha) = \alpha - \alpha\partial(\alpha) = 0$. Conversely, let $\partial(\alpha) \neq 1$ and there exists an element $x \neq 0$ such that $\pi(x) = 0$. Then we have

$$x = \alpha\partial x.$$

Appling ∂ we have

$$\partial x = \partial(\alpha)\partial x + \alpha\partial^2 x - \alpha\partial(\alpha)\partial^2 x.$$

Thus

$$(1 - \partial(\alpha))\partial x = \alpha(1 - \partial(\alpha))\partial^2 x.$$

Now the condition $\partial(\alpha) \neq 1$ implies

$$\partial x = \alpha\partial^2 x.$$

We use induction to obtain the identity

$$\partial^k x = \alpha\partial^{k+1} x$$

for any $k \geq 0$. This implies

$$x = \alpha^k\partial^k x$$

for all $k \geq 0$, i.e. $x \in \bigcap_k(\alpha^k R)$. However, it is possible only if $x = 0$ because of the assumptions on R. This proves Lemma 8. $\square$

Corollary 2. *Let ∂ be a divided difference operator. Then the multiplicative operator $\pi = 1-\alpha\partial$ is a projector ($\pi^2=\pi$) if and only if ∂ is a division operator by the element α.*

Proof. Let ∂ be a division operator by the element α. Then $\pi(\alpha)=0$ and $\pi^2 x = \pi(x-\alpha\partial x)=\pi x$. Conversely, let $\pi^2=\pi$. Then $\pi(\pi x - x)=0$ for any x and Lemma 8 gives that $\pi x = x$. This contradicts $\alpha\neq 0$. This proves the result. □

Lemma 9. *Let $\alpha\in R$ and let $\Pi_\alpha(R)$ be the set of multiplicative operators $\pi\in\Pi_\alpha(R)$ such that $\pi = 1-\alpha\partial$. Then the set $\Pi_\alpha(R)$ is a subsemigroup in the semigroup $\Pi(R)$ of all multiplicative operators $R\to R$ where operation in $\Pi(R)$ is given by $(\pi_1\circ\pi_2)(u)=\pi_1(\pi_2(u))$.*

Let $\pi_1,\pi_2\in\Pi_\alpha(R)$ and ∂_1, ∂_2 be the corresponding divided difference operators. Then ∂_1 and ∂_2 satisfy

$$\partial_1\circ\partial_2 = \partial_1+\partial_2-\partial_1(\alpha\partial_2), \qquad 1-\alpha\partial_i=\pi_i, \quad i=1,2. \tag{15}$$

Proof. By the definitions

$$(\pi_1\circ\pi_2)(u) = u-\alpha\partial_1\circ\partial_2(u).$$

Then

$$\begin{aligned}\alpha\partial_1\circ\partial_2(u) &= u-\pi_1\circ\pi_2(u) = u-\pi_1(\pi_2(u)) = u-\pi_1(u-\alpha\partial_2(u))\\ &= u-(u-\alpha\partial_2(u))+\alpha\partial_1[u-\alpha\partial_2(u)] = \alpha[\partial_1(u)+\partial_2(u)-\partial_1(\alpha\partial_2)(u)].\end{aligned}$$

Thus

$$\partial_1\circ\partial_2=\partial_1+\partial_2-\partial_1(\alpha\partial_2).$$

The result follows. □

Let us give some known examples of divided difference operators.

(1) Let $R=K[[x]]$, and $\alpha(x),\phi(x)\in R$ where $\alpha(0)=0$. For $p\in R$ let

$$\partial p(x) = \frac{p(x)-p(x-\alpha(x)\psi(x))}{\alpha(x)}.$$

Clearly ∂ always satisfies (12), i.e. this is a divided difference operator. We have

$$\pi p(x) = p(x)-\alpha\cdot\frac{p(x)-p(x-\alpha(x)\psi(x))}{\alpha} = p(x-\alpha(x)\psi(x)).$$

This is a usual translation of the argument.

(2) Let $R=K[[x,y]]$ and $\alpha = x-y$. Let

$$\partial p(x,y) = \frac{p(x,y)-p(y,x)}{x-y}\,. \tag{16}$$

Here $\pi(p(x,y))=p(y,x)$, i.e. $\pi^2=1$ and $\partial^2=0$.

For $R = K[[x_1, \dots, x_n]]$ one can define the operators ∂_{ij} by

$$\partial_{ij} p(x_1, \dots, x_n) = \frac{p(\dots x_i \dots x_j \dots) - p(\dots x_j \dots x_i \dots)}{x_i - x_j}.$$

These operators play an important part in the cohomology of the flag manifolds. They have been used often (under the same name "divided difference operators"). Their algebra is very beautiful.

Let $f(x, y) \in K[[x, y]]$ be a series defining some formal group over R. Let $\alpha = f(x, \bar{y})$ where $\bar{y}$ is the element inverse to y with respect to the formal group $f(x, y)$. Then the formula

$$\partial p(x, y) = \frac{p(x, y) - p(y, x)}{f(x, \bar{y})} \tag{17}$$

defines a divided difference operator in $R = K[[x, y]]$. Here $\pi(p(x, y)) = p(y, x)$, $\pi^2 = 1$, and $\partial^2 = \gamma\partial$ for some $\gamma \in K[[x, y]]$. For example, let $f(x, y) = x + y - axy$, $a \in K$. Then $\partial^2 = a\partial$.

The operators given by (17) with $f(x, y)$ being the formal group of geometric cobordism elements have interesting applications in the complex cobordism theory (see [18], [19]).

The examples (1) and (2) are particular cases of the following general construction.

Let $R = K[[x_1, \dots, x_n]]$ and $\alpha(x) \in R$ with $\alpha(0) = 0$. Put

$$\psi(x) = (\psi_1(x), \dots, \psi_n(x))$$

where $x = (x_1, \dots, x_n)$ and $\psi_k(x) \in R$, $k = 1, \dots, n$.

Now put

$$\partial p(x) = \frac{1}{\alpha(x)}\big(p(x) - p(x - \alpha(x)\psi(x))\big)$$

for any $p \in R$. We have

$$\pi p(x) = p(x - \alpha(x)\psi(x)).$$

For applications, the most interesting cases are when $\alpha(x) = \langle x, \xi\rangle = \sum_{i=1}^n x_i \xi^i$ and either

(i) $\psi(x) = \frac{\xi}{\langle \xi, \xi\rangle}$ where $\pi_\xi p(x) = p\big(x - \langle x, \xi\rangle \frac{\xi}{\langle \xi, \xi\rangle}\big)$ is a projector, $\pi_\xi^2 = \pi_\xi$, and $\partial = \partial_\xi$ is a division operator by $\langle x, \xi\rangle$, or

(ii) $\psi(x) = 2\frac{\xi}{\langle \xi, \xi\rangle}$ where $\pi_\xi p(x) = p\big(x - 2\langle x, \xi\rangle \frac{\xi}{\langle \xi, \xi\rangle}\big)$ satisfies the conditions $\pi_\xi^2 = 1$ and $\partial_\xi^2 = 0$.

These are the operators that are determined by reflections with respect to the hyperplanes $V = \{x \mid \langle x, \xi\rangle = 0\}$. They were used to define the Dunkl's operators, see [20].

Particular choices of a configuration space for the vectors $\xi_1, \dots, \xi_k$ lead to interesting algebras of the operators ∂_{ξ_l}, $l = 1, \dots, k$.

(3) Let $\phi : R \to R$ be a ring homomorphism. According to [21], a *derivation of the algebra* R is an operator $\delta : R \to R$, such that

$$\delta(ab) = \phi(a)\delta(b) + \delta(a)b. \tag{18}$$

Definition 4. An algebra over R additively isomorphic to $R[t]$ is called an *Ore extension* if $R \subset R[t]$ and

$$ta = \phi(a)t + \delta(a). \tag{19}$$

The operator ϕ from (19) is called *Ore's ϕ-derivation.*

One easily proves the following lemma.

Lemma 10. *A divided difference operator ∂ satisfying the identity* (12)

$$\partial(ab) = (\partial a)b + a(\partial b) - \alpha(\partial a)(\partial b)$$

where $\alpha \in R$ is an Ore's ϕ-derivation with $\phi = \pi = 1 - \alpha\partial : R \to R$.

Now we construct new K-linear product structures in the ring R.

Theorem 1. *Let $\alpha_1, \alpha_2 \in R$ and let ∂_1, ∂_2 be divided difference operators corresponding to the elements α_1, α_2. Let $\pi_1 = 1 - \alpha_1\partial_1$, $\pi_2 = 1 - \alpha_2\partial_2$ be corresponding multiplicative projectors. The operation*

$$\mu_1(x, y) = \pi_1(x)\pi_2(y) = xy - \alpha_1\partial_1(x)y - -\alpha_2\partial_2(y)x + \alpha_1\alpha_2\partial_1(x)\partial_2(y) \tag{20}$$

determines an associative product if and only if ∂_1 and ∂_2 are division operators and the multiplicative projectors π_1 and π_2 commute, i.e. $\pi_1\pi_2 = \pi_2\pi_1$. In particular, each division operator ∂ determines the associative product $\mu(x, y) = \pi(x)\pi(y)$ where $\pi = 1 - \alpha\partial$. The product $\mu_1(x, y)$ is commutative if and only if $\pi_1 = \pi_2$.

Proof. Let

$$x \circ y = \mu_1(x, y) = \pi_1(x)\pi_2(y).$$

Then

$$(x \circ y) \circ z = \pi_1(x \circ y)\pi_2(z) = \pi_1^2(x)\pi_1(\pi_2(y))\pi_2(z). \tag{21}$$

Assume that the divided difference operators ∂_1 and ∂_2 are division operators, i.e. $\partial_1(\alpha_1) = \partial_2(\alpha_2) = 1$. Thus $\pi_1^2 = \pi_1$, $\pi_2^2 = \pi_2$, and the identity (21) implies that the product μ_1 is associative.

Conversely, let μ_1 be associative. Then the condition

$$(x \circ 1) \circ 1 = x \circ (1 \circ 1)$$

implies

$$\pi_1^2(x)\pi_1(\pi_2(1))\pi_2(1) = \pi_1(x)\pi_2(\pi_1(1))\pi_2^2(1).$$

We have $\pi_1^2 = \pi_1$ since $\pi(1) = 1$ for any multiplicative operator π. Similarly, we have $\pi_2^2 = \pi_2$.

Now the condition $(1 \circ y) \circ 1 = 1 \circ (y \circ 1)$ implies that $\pi_1\pi_2 = \pi_2\pi_1$. We use Corollary 2 to conclude that the operators ∂_1 and ∂_2 corresponding to the projectors π_1 and π_2 are division operators.

Finally, if $\mu_1(x, y)$ is a commutative product then the condition $x \circ 1 = 1 \circ x$ implies that $\pi_1 = \pi_2$. This proves Theorem 1. $\square$

Theorem 2. *Let ∂ be a divided difference operator corresponding to an element $\alpha \in R$ and $\beta \in R$. The operation*

$$\mu_2(x,y) = xy + \beta\partial(x)\partial(y), \qquad \beta \in R, \tag{22}$$

determines an associative product in R if and only if one of the following conditions is satisfied:

(i) *the operator ∂ is a division operator and $\pi(\beta) = 0$ where $\pi = 1 - \alpha\partial$;*

(ii) *the operator ∂ is not a division operator and $\partial^2 x\partial y = \partial x\partial^2 y$ for any elements $x, y \in R$.*

In particular, the condition (ii) *is satisfied if $\partial^2 = \gamma\partial$.*

Furthermore, the condition (ii) *does not depend on a choice of β, i.e. β is a parameter of a deformation that connects the original product structure in the ring R to the product structure μ_2.*

Proof. Let

$$x \circ y = \mu_1(x,y) = xy + \beta\partial x\partial y.$$

Then

$$\begin{aligned}
(x \circ y) \circ z \ &= (xy + \beta\partial x\partial y)z + \beta\partial(xy + \beta\partial x\partial y)\partial z \\
&= xyz + \beta\big((\partial x\partial y)z + (\partial x)y\partial z + x\partial y\partial z\big) \\
&\quad -\beta\alpha\partial x\partial y\partial z + \beta\big(\partial\beta\partial x\partial y\partial z + \beta\partial(\partial x\partial y)\partial z - \alpha\partial\beta\partial(\partial x\partial y)\partial z\big).
\end{aligned}$$

Thus the product μ_2 is associative if and only if

$$(\beta - \alpha\partial\beta)\big(\partial(\partial x\partial y)\partial z - -\partial x\partial(\partial y\partial z)\big) = 0 \tag{23}$$

for all $x, y, z \in R$.

Let ∂ be a division operator, i.e. $\partial(\alpha) = 1$. Put $x = \alpha^2$, $y = z = \alpha$. Then we have $\partial(\alpha^2) = 2\alpha\partial\alpha - \alpha(\partial\alpha)^2 = \alpha$ whence $\partial^2(\alpha^2) = 1$. Using $\partial(1) = 0$ for the fixed values of x, y, z we see that (23) gives $\pi(\beta) = \beta - \alpha\partial\beta = 0$. This proves (i).

Now assume that ∂ is not a division operator. Then according to Lemma 8, $\pi(\beta) \neq 0$ if $\beta \neq 0$. The identity

$$\partial(\partial x\partial y) = \partial^2 x\partial y + \partial x\partial^2 y - \alpha\partial^2 x\partial y$$

shows that (23) is equivalent to

$$(\partial^2 x\partial z - \partial z\partial^2 z)(\partial y - \alpha\partial^2 y) = 0 \tag{24}$$

for all $x, y, z \in R$. We notice that $\partial y - \alpha\partial^2 y = \pi\partial y$. The operator ∂ is not a division operator by assumption thus $\pi\partial(y) = 0$ is equivalent to $\partial y = 0$ for all $y \in R$. This is impossible. It follows now that (24) is equivalent to $\partial^2 x\partial z = \partial z\partial^2 z$. This proves (ii) and the theorem. $\square$

Now we give one more construction of an associative product which comes from the complex cobordism theory (see [22]).

Theorem 3. *Let $\Pi : R \to R$ and $\delta : R \to R$ be linear operators such that*

(1) $\Pi^2 = \Pi$, $\delta\Pi = \delta$,

(2) $\delta(\Pi x \Pi y) = \delta x(\Pi y) + (\Pi x)\delta y - \alpha(\delta x)(\delta y)$,

(3) $\Pi((\Pi x)(\Pi y)) = (\Pi x)(\Pi y) + \beta(\delta x)(\delta y)$

for some elements $\alpha, \beta \in R$. Then the operation

$$\mu(x, y) = \Pi((\Pi x)(\Pi y)) = (\Pi x)(\Pi y) + \beta(\delta x)(\delta y)$$

is an associative product structure in the ring R.

Proof. We denote $\mu(x, y) = x * y$. Then the definitions imply

$$\begin{aligned}
(x * y) * z &= \Pi(x * y)\Pi z + \beta\delta(x * y)\delta z \\
&= (\Pi(\Pi x \Pi y))\Pi z + \beta\delta(\Pi((\Pi x)(\Pi y)))\delta z \\
&= \Pi x \Pi y \Pi z + \beta(\delta x)(\delta y)\Pi z + \beta[\delta x(\Pi y)\delta z + (\Pi x)\delta y \delta z - \alpha(\delta x)(\delta y)(\delta z)] \\
&= \Pi x \Pi y \Pi z + \beta[\delta x \delta y \Pi z + \delta x \Pi y \delta z + \Pi x \delta y \delta z] - \beta\alpha\delta x \delta y \delta z \\
&= x * (y * z).
\end{aligned}$$

The result follows. □

3 Examples from cobordism theory

Here we use the above results to construct new product structures in the complex cobordism theory.

First, we recall that all multiplicative operators in the algebra $A^U = (\Lambda S)^\wedge$ are completely determined by their action on geometric cobordism elements $x \in U^2(X)$, i.e. they all are given as series

$$\phi(x) = x + \sum_{i \geq 1} \phi_i x^{i+1}, \qquad \phi_i \in \Lambda. \tag{25}$$

A multiplicative operator ϕ given by a series $\phi(x)$ acts on an element $y \in U^*(X)$ as follows:

$$\phi(y) = y + \sum_{\deg w > 0} \phi_w s_w(y),$$

where $\phi_w = \phi_1^{k_1} \cdots \phi_l^{k_l}$ with $w = (k_1, \ldots, k_l)$.

Lemma 11. *A series $D\phi(x) = x + \sum_{i \geq 1} \phi_i x^{i+1}$ determines a multiplicative projector $\phi \in A^U$ (i.e. $\phi^2 = \phi$) if and only if $\phi(\phi_i) = \phi_i + \sum \phi_w s_w(\phi_i) = 0$ for all $i \geq 1$.*

Proof. Let $x \in U^2(X)$ be a geometric cobordism element. Then

$$\phi(\phi(x)) = \phi(x) + \sum \phi(\phi_i)\phi(x)^{i+1}.$$

Thus $\phi(\phi(x)) = \phi(x)$ if and only if $\sum_{i \geq 1} \phi(\phi_i) t^{i+1} = 0$ in the ring $\Lambda[[t]]$ with $t = \phi(x)$. The latter is equivalent to $\phi(\phi_i) = 0$, $i \geq 1$ and the result follows. □

Let $x \in U^2(X)$ be a geometric cobordism element. A multiplicative operator $\phi \in A^U$ with $\phi(x) = x + \sum_{i\geq 1} \phi_i x^{i+1}$, $\phi_i \in \Lambda$, is called *homogeneous* if $\deg \phi_i = -2i$ for all $i = 1, 2, \ldots$. The set of all homogeneous multiplicative operators form a semigroup. Clearly multiplicative projectors belong to this semigroup. This semigroup is a fundamental object in the complex cobordism theory.

Let m be an integer and $\mathbb{Z}_{(m)} = \mathbb{Z}[m^{-1}]$. The paper [16] contains the following result.

Lemma 12. *Let $\alpha \in \Lambda^{-2n}$ and $s_{(n)}\alpha = m \neq 0$, $m \in \mathbb{Z}$. An operator $\partial \in A^U$ is a division operator by an element α in the localized complex cobordism theory $U^*(X) \otimes \mathbb{Z}_{(m)}$ (so that $\pi = 1 - \alpha\partial$ is a homogeneous multiplicative operator) if and only if the value of ∂ on any geometric cobordism element x is given by*

$$\partial x = \frac{1}{m} x^{n+1} + \sum_{i\geq 1} a_i x^{n+i+1}, \qquad a_i \in \Lambda^{-2i} \otimes \mathbb{Z}_{(m)},$$

where (a_i) are free parameters of ∂.

Proof. According to the above algebraic results, it is enough to verify that $\pi(\alpha) = 0$. Indeed, we have: $\pi(x) = x - \alpha x^n \left(\frac{1}{m}x + \sum_{i\geq 1} a_i x^{i+1}\right)$ where x is a geometric cobordism element. Thus

$$\pi(y) = y - \frac{1}{m}\alpha \cdot s_{(n)} y + (\text{terms containing } s_w(y) \text{ with } \deg w > 2n).$$

We conclude that $\pi(\alpha) = \alpha - \alpha \frac{1}{m} s_{(n)}(\alpha) = 0$. The result follows. $\square$

Lemma 12 may be used to describe all pairs (∂_1, ∂_2) of division operators such that the corresponding multiplicative operators π_1 and π_2 are homogeneous. Thus according to Theorem 1, we obtain the set of multiplicative associative product structures (in the complex cobordism theory) of the form $\mu_1(x, y) = \pi_1(x)\pi_2(y)$.

Moreover, let ∂ be a division operator from Lemma 12, $\pi = 1 - \alpha\partial$, and $\beta \in \operatorname{Ker}(\pi)$. Then according to Theorem 2 (ii), we obtain an associative product structure $\mu_2(x, y) = xy + \beta \partial x \partial y$.

The following result provides a construction of associative product structures based on Theorem 2 (ii).

Lemma 13. *Let $\alpha \in \Lambda^{-2n}$ and $s_{(n)}(\alpha) = 2n$. Then any multiplicative homogeneous operation $\pi \in A^U \otimes \mathbb{Z}_{(n)}$ given by*

$$\pi(x) = x - \alpha\partial x = \frac{x}{\sqrt[n]{1 + \alpha x^n}} \tag{26}$$

on a geometric cobordism element $x \in U^2(X)$ determines the divided difference operator ∂ such that $\partial^2 = 0$.

Proof. According to Lemma 6, it is enough to verify that

$$\pi(\alpha) = -\alpha \quad \text{and} \quad \pi^2 = 1.$$

We have

$$\pi(x) = x - \frac{1}{n}\alpha x^{n+1} + O(x^{2n+1}).$$

Thus

$$\pi(y) = y - \frac{1}{n}\alpha \cdot s_{(n)}y + (\text{terms containing } s_w(y) \text{ with } \deg w > 2n).$$

Then it follows that $\pi(\alpha) = \alpha - \frac{1}{n}\alpha \cdot 2n = -\alpha$. The action of π^2 is determined by its value on a geometric cobordism element since π^2 is a multiplicative operator. We have

$$\pi^2(x) = \pi\left(\frac{x}{\sqrt[n]{1+\alpha x^n}}\right) = \frac{\pi(x)}{\sqrt[n]{1+\pi(\alpha)\pi(x)^n}} = \frac{x}{\sqrt[n]{1+\alpha x^n}} \cdot \frac{1}{\sqrt[n]{1-\alpha\dfrac{x^n}{1+\alpha x^n}}} = x.$$

Thus $\pi^2(x) = x$ whence $\pi^2 = 1$. The result follows. □

Example. Let $[\mathbb{C}P^m]$ be the cobordism class of $\mathbb{C}P^m$. Then $s_{(m)}[\mathbb{C}P^m] = -(m+1)$. Hence we can choose α equal to $-[\mathbb{C}P^{2n-1}]$ to obtain the divided difference operator ∂ on $U^*(X) \otimes \mathbb{Z}_{(n)}$ (according to Lemma 13). Here ∂ satisfies $\partial^2 = 0$ which gives a new associative product structure depending on free parameter β. For instance, the complex projective plane $\mathbb{C}P^1$ gives such a product on $U^*(X)$.

In conclusion, we describe particular example of an associative product on $U^*(X)$ based on the construction from Theorem 3.

Following Conner and Floyd, we define the additive projector

$$\Pi : U^*(X) \to U^*(X)$$

which is completely determined by the following action on the complex cobordism classes.

Let $\xi \to \mathbb{C}P^1$ be a cononical line bundle and M^{2n} a closed U-manifold. Let τ be its stable complex tangent bundle. By definition, $\Pi([M^{2n}])$ is a cobordism class of the manifold $i : \widehat{M}^{2n} \subset M^{2n} \times \mathbb{C}P^1$ with the normal bundle $i^*((\det \tau) \otimes \xi)$ where $\det \tau$ is the determinant bundle.

The action of the projector $\Pi \in A^U$ on the Thom class x_n (of a complex n-dimensional bundle $\eta \to X$) is given by

$$\Pi(x_n) = x_n + \sum_{i \geq 2} \alpha_{i1} \partial_i x_n$$

where $\alpha_{i1} \in \Lambda^{-2i}$ is the coefficient with $x^i y$ in the formal group $f(x,y)$ of geometric cobordism elements x, y. Here $\partial_i \in A^U$ are the projectors acting on the Thom class x_n as

$$\partial_i x_n = x_n c_1(\overline{\det \eta})$$

where c_1 is the first Chern class in complex cobordism and "$\overline{}$" is the complex conjugation.

Let $\delta = \partial_1 \in A^U$. Then one can verify that the pair (Π, δ) satisfies the conditions of Theorem 3 (see [22], [23] for details). Thus according to Theorem 3, we obtain the associative product given by

$$x * y = \Pi(\Pi x \Pi y).$$

This product structure is crucial to describe a ring structure of the cobordism ring of SU-manifolds (these manifolds are called sometimes Calabi-Yau manifolds).

References

[1] Novikov, S. P. Topology I, Springer-Verlag Berlin, 1996. (Encyclopaedia of Math. Sci. Vol. 12.)

[2] Milnor J. W. On the cobordism ring Ω^* and a complex analogue. Part I, Amer. J. Math. 82, (1960) 505–521.

[3] Novikov, S. P. Some problems in the topology of manifolds connected with the theory of Thom spaces. Soviet Math. Dokl. 1, (1960) 717–720.

[4] Novikov, S. P. Homotopy properties of Thom complexes. Mat. Sb. 57, (1962) 407–442.

[5] Novikov, S. P. Methods of algebraic topology from the point of view of cobordism theory. Izv. Akad. Nauk SSSR Ser. Mat. 31, (1967) 855–951.

[6] Landweber, P. S. Cobordism operations and Hopf algebras. Trans. Amer. Math. Soc. 129, (1967) 94–110.

[7] Buchstaber, V. M.; Shokurov, A. V. The Landweber-Novikov algebra and formal vector fields on the line. Funkt. Anal. i Prilozhen. 12, (1978) 1–11, 96.

[8] Novikov, S. P. Various doublings of Hopf algebras. Algebras of operators on quantum groups, complex cobordisms. Russian Math. Surveys 47, (1992) 198–199.

[9] Buchstaber, V. M. Semigroups of maps into groups, operator doubles, and complex cobordisms. Topics in topology and mathematical physics, 9–31, Amer. Math. Soc. Transl. Ser. 2, 170, Amer. Math. Soc., Providence, RI, 1995.

[10] Buchstaber, V. M. The Chern-Dold character in cobordisms. I. Mat. Sb. 83, (1970) 575–595.

[11] Buchstaber, V. M.; Novikov, S. P. Formal groups, power systems and Adams operators. Mat. Sb. 84, (1971), 81–118.

[12] Buchstaber, V. M. Characteristic classes in cobordism and topological applications of the theory of single-valued and two-valued formal groups. Current problems in mathematics, Vol. 10 pp. 5–178, VINITI, Moscow, 1978.

[13] Buchstaber, V. M.; Kholodov, A. N. Topological constructions connected with many-valued formal groups. Izv. Akad. Nauk SSSR Ser. Mat. 46, (1982), 3–27.

[14] Kholodov, A. N. The algebraic theory of multivalued formal groups. Mat. Sb. 114, (1981), 299–321.

[15] Buchstaber, V. M.; Rees, E. G. Multivalued groups, their representations and Hopf algebras. Transform. Groups 2, (1997) 325–349.

[16] Buchstaber, V. M. Cohomology operations and the formal group law in cobordism. Characteristic classes in cobordism theory, p. 227–248. Appendix to the Russian translation of book *Snaith, Victor P. Algebraic cobordism and K-theory. Mem. Amer. Math. Soc. 21 (1979), no. 221.*

[17] Bernshtein, I. N.; Gelfand, I. M.; Gelfand, S. I. Schubert cells, and the cohomology of the spaces G/P. (Russian) Uspehi Mat. Nauk 28, (1973) 3–26.

[18] Bressler, P.; Evens, S. The Schubert calculus, braid relations, and generalized cohomology. Trans. Amer. Math. Soc. 317, (1990) 799–811.

[19] Feldman K. E. Equivariant analogue of the Poincare-Birkhoff theorem, Notes of LOMI, 247, (2000) 160-174.

[20] Dunkl, C. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. 311 (1989) 167–183.

[21] Ore O. Theory of non-commutative polynomials, Ann. of Math. 34, (1933) 480–508.

[22] Buchstaber, V. M. Classification of two-valued formal groups. (Russian) Uspehi Mat. Nauk 28, (1973) 173–174.

[23] Stong, Robert E. Notes on cobordism theory. Mathematical notes Princeton University Press, Princeton, 1968.